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ON THE GRADED IDENTITIES OF $M_{1,1}(E)$

BY

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ABSTRACT

We determine the $S_n \times S_m$-cocharacter $\chi_{n,m}$ of the algebra $M_{1,1}(E)$ and prove that the T_2-ideal of its graded identities is generated by the polynomials $y_1y_2 - y_2y_1$ and $z_1z_2z_3 + z_3z_2z_1$.

$\mathbb{Z}_2$-graded algebras and their graded identities have been used in [1] and [4] to study the structure of varieties of associative algebras over a field F of characteristic zero.

In [2], Berele defined a $S_n \times S_m$-cocharacter $\chi_{n,m}$ for $\mathbb{Z}_2$-graded algebras and he related this cocharacter to the ordinary S_{n+m}-cocharacter (for P.I.-algebras).

Moreover, in [7], Regev used these last results to obtain a description of codimensions of the algebras $M_{k,l}(E)$, which play an important role in the theory of P.I.-algebras. More precisely, as proved by Kemer in [2], any non-trivial prime variety is generated by any one of the algebras, $M_n(F)$, $M_n(E)$, $M_{k,l}(E)$ which are defined as follows.

Let A be an algebra over F and let $M_n(A)$ denote the $n \times n$ matrices over A. Let E be the Grassman algebra of a countable dimensional vector space over F. By considering the length of the basis elements of E we have that $E = E_0 \oplus E_1$, where E_0 is the vector space spanned by the elements of even length and E_1 is spanned by the elements of odd length. Given $k, l \geq 0$ we denote by $M_{k,l}(E)$ the following subalgebras of $M_{k+1}(E)$:

$$M_{k,l}(E) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{array}{l} A \in M_k(E_0),\ D \in M_l(E_0),\ B, C \text{ are respectively} \\ k \times l \text{ and } l \times k \text{ matrices, both with entries in } E_1 \end{array} \right\}.$$

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In this paper we will determine the $S_n \times S_m$-cocharacter $\chi_{n,m}$ of the algebra $M_{1,1}(E)$ and prove that the T_2-ideal of its graded identities is generated by the polynomials $[y_1, y_2] = y_1y_2 - y_2y_1$ and $z_1z_2z_3 + z_3z_2z_1$.

As a consequence of our theorem and using a result of Popov [5], we show that $M_{1,1}(E)$ is equivalent to $E \otimes E$, that is they satisfy the same polynomial identities.

We remark that this last result was obtained by Kemer in [4], as a consequence of his structure theory for varieties of algebras, but our proof does not make use of the above structure theory (another proof of this result is contained in [8]).

Definitions and preliminary results

Following the definitions of Kemer, an algebra A is $\mathbb{Z}_2$-graded if $A = A_0 + A_1$, where A_0,A_1 are subspaces of A satisfying:

$$A_0A_0 + A_1A_1 \subseteq A_0 \text{ and } A_0A_1 + A_1A_0 \subseteq A_1.$$

Now, let $F\{X\}$ be the free algebra over the field F generated by a countable set X. As in [4] we represent X in the form $X = Y \cup Z$ where Y and Z are countable disjoint subsets of X.

$\mathcal{F}_0$ will denote the subspace of $F\{X\}$ generated by the monomials of even degree with respect to Z; similarly $\mathcal{F}_1$ will denote the subspace of $F\{X\}$ generated by the monomials of odd degree with respect to Z.

An ideal I of $F\{X\}$ is a T_2-ideal if it is invariant under all F-endomorphisms η of $F\{X\}$ such that $\eta(\mathcal{F}_0) \subseteq \mathcal{F}_0$ and $\eta(\mathcal{F}_1) \subseteq \mathcal{F}_1$.

A polynomial $f(y_1, \dots, y_n, z_1, \dots, z_m)$ is a graded identity of a $\mathbb{Z}_2$-graded algebra $A = A_0 + A_1$ if $f(a_1, \dots, a_n, b_1, \dots, b_m) = 0$ for all $a_1, \dots, a_n \in A_0$ and $b_1, \dots, b_m \in A_1$.

The set $I = T_2(A)$ of all graded identities of A is a T_2-ideal of $F\{X\}$.

Let $V_{n,m}$ be the space of all multilinear polynomials of degree $n + m$ in the variables $y_1, \dots, y_n, z_1, \dots, z_m$ and let for a T_2-ideal $I, I_{n,m} = I \cap V_{n,m}$. Clearly, $I_{n,m}$ becomes a $S_n \times S_m$-submodule of $V_{n,m}$ if, as usually, we define

$$(\sigma, \pi)f(y_1, \dots, y_n, z_1, \dots, z_m) = f(y_{\sigma(1)}, \dots, y_{\sigma(n)}, z_{\pi(1)}, \dots, z_{\pi(m)}),$$

for all $(\sigma, \pi) \in S_n \times S_m$ and $f(y_1, \dots, y_n, z_1, \dots, z_m) \in V_{n,m}$.

We denote by $\chi_{n,m}(I)$ $(\chi_{m,n}(A))$ the $S_n \times S_m$-character of the quotient module $V_{n,m}/I_{n,m}$, and by $c_{n,m}(I)$ $(c_{n,m}(A))$ its dimension over F.

It follows from the theory of representations of the symmetric group that

$$\chi_{n,m}(I) = \chi_{n,m}(A) = \sum_{\substack{\lambda \vdash n \\ \mu \vdash m}} m_{\lambda,\mu}[\lambda] \otimes [\mu],$$

where $[\lambda] \otimes [\mu]$ denotes the irreducible $S_n \times S_m$-character given by the tensor product of the irreducible characters $[\lambda]$, $[\mu]$ corresponding to the partitions λ, μ of n and m respectively (see [3]).

Moreover, $m_{\lambda,\mu} \neq 0$ if and only if there exist a λ-tableau T_1, a μ-tableau T_2 and some monomial $M(y_1, \ldots, y_n, z_1, \ldots, z_m)$ of $V_{n,m}$ such that the polynomial $e_{T_1} e_{T_2} M(y_1, \ldots, y_n, z_1, \ldots, z_m)$ is not a graded identity of A.

Here e_{T_i} $(i = 1, 2)$ denotes the essential idempotent element of FS_n (FS_m) given by $e_{T_i} = \sum_{\sigma \in R_{T_i}} \sum_{\pi \in C_{T_i}} (-1)^\pi \sigma\pi$, where R_{T_i}, C_{T_i} are the subgroups of S_n (S_m) fixing respectively the rows and the columns of T_i.

The graded identities of $M_2(F)$

We consider the algebra $A = M_2(F)$ with the non-trivial grading

$$A_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a, d \in F \right\}, \quad A_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \middle| b, c \in F \right\},$$

and let I be the T_2-ideal of its graded identities.

We start by considering a suitable subset of $V_{n,m}$ which leads to a basis of $V_{n,m}/I_{n,m}$. More precisely we have the following definition.

Definition 1: For $m > 0$, let $(j) = \{j_1, j_2, \ldots, j_{[m/2]}\}$ be a subset of $\{1, \ldots, m\}$ of order $[m/2]$ and let $(i) = \{i_1, i_2, \ldots\}$ be its complement in $\{1, \ldots, m\}$. Moreover, for $q = 0, 1, \ldots, n$ let $(t) = \{t_1, \ldots, t_q\}$ be a subset of $\{1, \ldots, n\}$ of order q, and let $(s) = \{s_1, s_2, \ldots\}$ be its complement in $\{1, \ldots, n\}$. ∎

We separately write in increasing order all integers occurring in the distinct sets (i), (j), (t), (s) and we put

$$\begin{aligned} M_{(t),(j)} &= M_{(t),(j)}(y_1, \ldots, y_n, z_1, \ldots, z_m) \\ &= \begin{cases} y_{t_1} y_{t_2} \cdots y_{t_q} z_{i_1} y_{s_1} \cdots y_{s_{n-q}} z_{j_1} z_{i_2} z_{j_2} \cdots z_{i_{[m/2]}} z_{j_{[m/2]}} & m \text{ even} \\ y_{t_1} y_{t_2} \cdots y_{t_q} z_{i_1} y_{s_1} \cdots y_{s_{n-q}} z_{j_1} z_{i_2} z_{j_2} \cdots z_{i_{[m/2]+1}} & m \text{ odd} \end{cases} \end{aligned}$$

We have:

LEMMA 1: *The $2^n\binom{m}{[\frac{m}{2}]}$ monomials $M_{(t),(j)}$ are linearly independent modulo $I_{n,m}$.*

Proof: We assume m even (the proof of the odd case is very similar).

Let $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ be the standard basis of $M_2(F)$ and

$$z_i \longmapsto \bar{z}_i = a_i e_{12} + b_i e_{21}, \qquad y_i \longmapsto \bar{y}_i = \alpha_i e_{11} + \beta_i e_{22}$$

be the most general graded substitution.

For each monomial, we have

$$\begin{aligned} M_{(t),(j)}(\bar{y}_1, \ldots, \bar{y}_n, \bar{z}_1, \ldots, \bar{z}_m) = & \\ & \alpha_{t_1} \cdots \alpha_{t_q} \beta_{s_1} \cdots \beta_{s_{n-q}} a_{i_1} b_{j_1} a_{i_2} b_{j_2} \cdots a_{i_{[m/2]}} b_{j_{[m/2]}} e_{11} \\ & + \beta_{t_1} \cdots \beta_{t_q} \alpha_{s_1} \cdots \alpha_{s_{n-q}} b_{i_1} a_{j_1} b_{i_2} a_{j_2} \cdots b_{i_{[m/2]}} a_{j_{[m/2]}} e_{22}. \end{aligned}$$

Let $f = \sum_{(t),(j)} A_{(t),(j)} M_{(t),(j)}(y_1, \ldots, y_n, z_1, \ldots, z_m)$ be a graded identity of $M_2(F)$; for a fixed $(j) = \{j_1, j_2, \ldots\}$ we specialize in the substitution

$$a_{i_1} = \cdots = a_{i_{[m/2]}} = b_{j_1} = \cdots = b_{j_{[m/2]}} = 1$$

and

$$b_{i_1} = \cdots = b_{i_{[m/2]}} = a_{j_1} = \cdots = a_{j_{[m/2]}} = 0.$$

Hence, for all $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in F$, we have

$$0 = f(\bar{y}_1, \ldots, \bar{y}_n, \bar{z}_1, \ldots, \bar{z}_m) = \sum_{(t)} A_{(t),(j)} \alpha_{t_1} \cdots \alpha_{t_q} \beta_{s_1} \cdots \beta_{s_{n-q}} e_{11}.$$

Since the characteristic of F is zero it follows that $A_{(t),(j)} = 0$ for all (t), and this proves the Lemma. ∎

Now we can prove our first result about the graded identities of $M_2(F)$.

LEMMA 2: *I is the T_2-ideal generated by $y_1y_2 - y_2y_1$ and $z_1z_2z_3 - z_3z_2z_1$. Moreover $c_{n,0}(I) = 1$ and $c_{n,m}(I) = 2^n \binom{m}{[\frac{m}{2}]}$ for $n \geq 0$, $m > 0$.*

Proof: Let J be the T_2-ideal of $F\{X\}$ generated by $y_1y_2 - y_2y_1$ and $z_1z_2z_3 - z_3z_2z_1$. Since these polynomials are graded identities of $M_2(F)$ it follows that $J \subseteq I$, and so $c_{n,m}(J) \geq c_{n,m}(I)$, for all $n, m \geq 0$.

We remark that $I_{n,0} = J_{n,0}$ as A_0 is commutative and non-nilpotent. Therefore $V_{n,0}/I_{n,0}$ is the one-dimensional S_n module with character $[(n)]$.

Now we assume $m > 0$.

Since J is a T_2-ideal then $y_iM - My_i \in J$, for each monomial M having even degree in the z_j's. This implies that $V_{n,m}$ is spanned modulo $J_{n,m}$ by the monomials $u_0(y)z_{i_1}u_1(y)z_{i_2}\cdots z_{i_m}$, where $u_0(y)$ and $u_1(y)$ are monomials (eventually 1) in $y_1,\ldots,y_n$ in which the y_i's occur in increasing order.

Now we consider the right action of the symmetric group S_m on the vector space $V_{0,m}$ of all multilinear polynomials in $z_1,\ldots,z_m$. This action is defined as follows:

$$(z_{i_1}\cdots z_{i_m})\sigma^{-1} = z_{i_\sigma(1)}\cdots z_{i_{\sigma(m)}},$$

that is σ acts on the monomial $M = M(z_1,\ldots,z_m)$ by changing the order of the z_i's. Hence the polynomials

$$z_{t_1}\cdots z_{t_m}(1-(i,i+2)) = z_{t_1}\cdots z_{t_i}z_{t_{i+1}}z_{t_{i+2}}\cdots z_{t_m} - z_{t_1}\cdots z_{t_{i+2}}z_{t_{i+1}}z_{t_i}\cdots z_{t_m}$$

are in the T_2-ideal J, and so $z_{t_1}\cdots z_{t_m} \equiv z_{t_1}\cdots z_{t_m}(i,i+2)$ mod. J, for all i.

Let G be the subgroup of S_m generated by the involutions $(i,i+2)$, with $i = 1,\ldots,m-2$, then $z_{t_1}\cdots z_{t_m}g \equiv z_{t_1}\cdots z_{t_m}$ mod. J, for all $g \in G$. Since G is the direct product of the symmetric groups G_1 and G_2 acting respectively on odd and even digits, then, for each monomial $M = M(z_1,\ldots,z_m)$ of $V_{0,m}$, we can apply a suitable $g \in G$ and separately write the z_i's occurring in the even and odd position in M in increasing order; moreover $M - Mg \in J$.

Since J is a T_2-ideal, it follows from the previous argument that $V_{n,m}$ is spanned modulo $J_{n,m}$ by the monomials $M_{(t),(j)}$ which were defined above. Therefore $c_{n,m}(J) \le 2^n\binom{m}{[\frac{m}{2}]}$.

On the other hand by Lemma 1, $2^n\binom{m}{[\frac{m}{2}]} \le c_{n,m}(I)$.

This inequality together with the previous implies

$$c_{n,m}(I) = c_{n,m}(J) = 2^n\binom{m}{[\frac{m}{2}]}$$

and moreover, as $J \subseteq I$, $I_{n,m} = J_{n,m}$ for all $n, m \ge 0$. Since I and J are T_2-ideals of $F\{X\}$ and char. $F = 0$, this suffices to conclude the proof. ∎

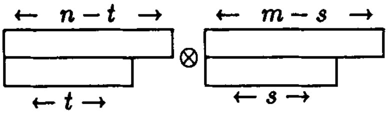
We remark that this last result together with Lemma 1 shows that the set $\{M_{(t),(j)} + I_{n,m}\}$ is a basis of $V_{n,m}/I_{n,m}$ for all $n \ge 0$ and $m > 0$.

Next we will examine more closely the $S_n \times S_m$-structure of $V_{n,m}/I_{n,m}$.

Let q be a fixed integer with $0 \le q \le n$, assume $m > 0$; and let W_q be the $S_n \times S_m$-submodule of $V_{n,m}/I_{n,m}$ generated by $y_1\cdots y_qz_1y_{q+1}\cdots y_nz_2z_3\cdots z_m + I_{n,m}$.

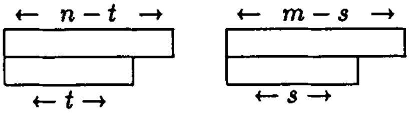
We identify the characters $[\lambda],[\mu]$ associated to the partitions λ,μ with the corresponding Young diagrams; with this notation we have

LEMMA 3: *For all s,t with $0 \le t \le q^* = \min\{q, n-q\}$ and $0 \le s \le [m/2]$, the irreducible $S_n \times S_m$-character*



is a component of $\chi_{n,m}(W_q)$.

Proof: In order to prove the Lemma it suffices to exhibit for each couple of diagrams



a couple of tableaux T_1,T_2 such that the corresponding element

$$(e_{T_1} e_{T_2}) y_1 \cdots y_q z_1 y_{q+1} \cdots y_n z_2 z_3 \cdots z_m \text{ of } V_{n,m}$$

is not a graded identity of $M_2(F)$ (see introduction).

This is obtained in the following manner.

Suppose, for simplicity of notations, $q^* = q$ and m even (all other cases can be treated in a similar way). For $t \le q$, $s \le [m/2]$ we consider the following Young tableaux

$$T_1 = \underset{\longleftarrow \qquad t \qquad \longrightarrow}{\overset{\longleftarrow \qquad\qquad\qquad\qquad n-t \qquad\qquad\qquad\qquad \longrightarrow}{\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline q+1 & q+2 & \cdots & q+t & t+1 & \cdots & q & q+t+1 & \cdots & n \\ \hline 1 & 2 & \cdots & t \\ \cline{1-4} \end{array}}}$$

$$T_2 = \underset{\longleftarrow \quad s \quad \longrightarrow}{\overset{\longleftarrow \qquad\qquad m-s \qquad\qquad \longrightarrow}{\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 3 & \cdots & 2s-1 & 2s+1 & \cdots & m \\ \hline 2 & 4 & \cdots & 2s \\ \cline{1-4} \end{array}}}$$

and let $R^{T_i} = \sum_{\sigma \in R_{T_i}} \sigma$, $C^{T_i} = \sum_{\pi \in C_{T_i}} (-1)^{\pi} \pi$, for $i = 1,2$.

Let

$$M = M(y,z) = y_1 \cdots y_q z_1 y_{q+1} \cdots y_n z_2 z_3 \cdots z_m,$$

$$M_1 = M_1(y,z) = y_1 \cdots y_q z_1 y_{q+1} \cdots y_n z_2,$$

$$M_2 = M_2(y,z) = y_1 \cdots y_q z_2 y_{q+1} \cdots y_n z_1,$$

and

$$f(z) = [z_3, z_4] \cdots [z_{2s-1}, z_{2s}] z_{2s+1} \cdots z_m,$$

then

$$e_{T_1} e_{T_2} M = R^{T_1} R^{T_2} C^{T_1} C^{T_2} M,$$

moreover

$$C^{T_2} M = M_1(y, z) f(z) - M_2(y, z) f(z).$$

Now let $\pi \in C_{T_1}$ and let $(i) = \{i_1, \dots, i_r\} \subseteq \{1, \dots, t\}$ be the ordered set of integers of the second row of T_1 which are moved by π; that is

$$\pi = (i_1, q + i_1)(i_2, q + i_2) \cdots (i_r, q + i_r).$$

Then $(-1)^\pi \pi M_1(y, z) f(z)$ is congruent modulo $I_{n,m}$ to

$$(-1)^r y_{j_1} \cdots y_{j_v} y_{t+1} \cdots y_q y_{q+i_1} \cdots y_{q+i_r} z_1 y_{i_1} \cdots y_{i_r}$$
$$y_{q+j_1} \cdots y_{q+j_v} y_{q+t+1} \cdots y_n z_2 f(z)$$

where $\{j_1, \dots j_v\} = \{1, \dots, t\} - \{i_1, \dots, i_r\}$ and $j_1 < \cdots < j_v$; in the same way $(-1)^\pi \pi M_2(y, z) f(z)$ is congruent modulo $I_{n,m}$ to

$$(-1)^r y_{j_1} \cdots y_{j_v} y_{t+1} \cdots y_q y_{q+i_1} \cdots y_{q+i_r} z_2 y_{i_1} \cdots y_{i_r}$$
$$y_{q+j_1} \cdots y_{q+j_v} y_{q+t+1} \cdots y_n z_1 f(z).$$

Hence $C^{T_1} C^{T_2} M = C^{T_1}(M_1 - M_2) f(z)$ is congruent, modulo $I_{n,m}$, to

$$g(y_1, \dots, y_n, z_1, \dots, z_m) = \sum_{r=0}^{t} \sum_{1 \leq i_1 < \cdots < i_r \leq t} (-1)^r$$
$$\big((y_{j_1} \cdots y_{j_v} y_{t+1} \cdots y_q y_{q+i_1} \cdots y_{q+i_r} z_1 y_{i_1} \cdots y_{i_r} y_{q+j_1} \cdots y_{q+j_v} y_{q+t+1} \cdots y_n z_2)$$
$$-(y_{j_1} \cdots y_{j_v} y_{t+1} \cdots y_q y_{q+i_1} \cdots y_{q+i_r} z_2 y_{i_1} \cdots y_{i_r} y_{q+j_1} \cdots y_{q+j_s} y_{q+t+1} \cdots y_n z_1)\big)$$
$$\cdot f(z)$$

where, as above,

$$\{j_1, \dots, j_v\} = \{1, \dots, t\} - \{i_1, \dots, i_r\} \quad \text{and} \quad j_1 < \cdots < j_v.$$

Now in $g(y_1, \dots, y_n, z_1, \dots, z_m)$ we identify $y_1 = y_2 = \cdots = y_t$, $y_{t+1} = y_{t+2} = \cdots = y_n$, $z_2 = z_4 = \cdots = z_{2s}$ and also $z_1 = z_3 = \cdots = z_{2s-1} = z_{2s+1} = z_{2s+2} = \cdots = z_m$.

In this way we obtain the polynomial

$$g_{T_1,T_2}(y_1,y_{t+1},z_1,z_2)=\sum_{r=0}^{t}\binom{t}{r}(-1)^r\big(y_1^{t-r}y_{t+1}^{q+r-t}z_1y_1^ry_{t+1}^{n-q-r}z_2 - y_1^{t-r}y_{t+1}^{q+r-t}z_2y_1^ry_{t+1}^{n-q-r}z_1\big)[z_1,z_2]^{s-1}z_1^{m-2s}.$$

A standard argument shows that

$$g_{T_1,T_2}(y_1,y_{t+1},z_1,z_2)\in I \Leftrightarrow (e_{T_1}e_{T_2})y_1\cdots y_qz_1y_{q+1}\cdots y_nz_2z_3\cdots z_m\in I_{n,m}$$

(see [6, sections 1,3] for details).

Let $y_1\mapsto\bar y_1=\alpha_1e_{11}+\beta_1e_{22}$, $y_{t+1}\mapsto\bar y_{t+1}=\alpha_2e_{11}+\beta_2e_{22}$, $z_i\mapsto\bar z_i$ be a graded substitution of the variables with elements of $M_2(F)$. One has

$$g_{T_1,T_2}(\bar y_1,y_{t+1},\bar z_1,\bar z_2)=\sum_{r=0}^{t}\binom{t}{r}(-1)^r\big(\alpha_1^{t-r}\alpha_2^{q+r-t}\beta_1^r\beta_2^{n-q-r}e_{11} + \beta_1^{t-r}\beta_2^{q+r-t}\alpha_1^r\alpha_2^{n-q-r}e_{22}\big)\cdot[\bar z_1,\bar z_2]^s\bar z_1^{m-2s}.$$

If $\bar z_1=e_{12}+e_{21}$ and $\bar z_2=e_{21}$, then $[\bar z_1,\bar z_2]^s\bar z_1^{m-2s}$ is an invertible matrix of $M_2(F)$, hence $g_{T_1,T_2}(\bar y_1,\bar y_{t+1},\bar z_1,\bar z_2)=0$ implies

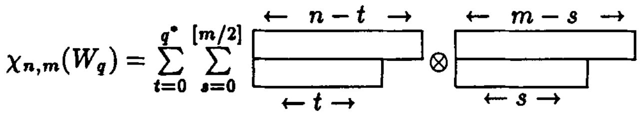
$$\sum_{r=0}^{t}\binom{t}{r}(-1)^r\alpha_1^{t-r}\alpha_2^{q+r-t}\beta_1^r\beta_2^{n-q-r}=0 \qquad \text{for all } \alpha_i,\beta_i\in F.$$

Since char $F=0$ this is impossible. Hence

$$g_{T_1,T_2}(y_1,y_{t+1},z_1,z_2)\notin I$$

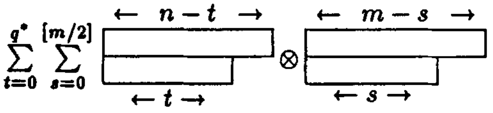
and this completes the proof. ∎

LEMMA 4:

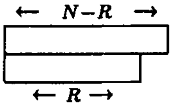


where $q^*=\min\{q,n-q\}$.

Proof: Let d be the degree of the $S_n\times S_m$-representation associated to



By the hook formula (see [3]) the dimension of the irreducible representation associated to Young diagram



is $\binom{N+1}{R}\left(1-\frac{2R}{N+1}\right)$.

Now by an easy induction on p we have $\sum_{R=0}^{p}\binom{N+1}{R}\left(1-\frac{2R}{N+1}\right)=\binom{N}{p}$ for all $p=0,\ldots,N$; and so $d=\binom{n}{q^*}\binom{m}{[m/2]}$.

On the other hand W_q is spanned over F by the elements

$$y_{\sigma(1)}\cdots y_{\sigma(q)}z_{\pi(1)}y_{\sigma(q+1)}\cdots y_{\sigma(n)}z_{\pi(2)}\cdots z_{\pi(m)}+I_{n,m}, \qquad (\sigma,\pi)\in S_n\times S_m.$$

Moreover, as shown in the proof of Lemma 2, each of these is equal to one of the elements $M_{(t),(j)}(y_1,\ldots,y_n,z_1,\ldots,z_m)+I_{n,m}$, with $(t)=\{t_1,\ldots,t_q\}$ and q is fixed.

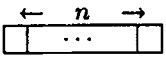
Since $M_{(t),(j)}(y_1,\ldots,y_n,z_1,\ldots,z_m)+I_{n,m}$ are in the basis of $V_{n,m}/I_{n,m}$ then $\dim W_q=\binom{n}{q}\binom{m}{[m/2]}$; moreover, since $\binom{n}{q}=\binom{n}{n-q}$ and $q^*=\min\{q,n-q\}$, one has $\dim W_q=d$.

Therefore the result follows by Lemma 3. ∎

Notice that the basis of $V_{n,m}/I_{n,m}$ given in the proof of Lemma 1 and Lemma 2 splits in the basis of the submodules W_q which we defined above, for $q=0,1,\ldots,n$; therefore $V_{n,m}/I_{n,m}=\oplus W_q$ and we have:

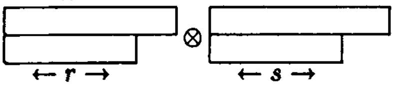
LEMMA 5: *Let $A=M_2(F)$ with the non-trivial grading. Then*

$$\chi_{n,0}(A)=$$

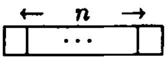


and, for $m>0$,

$$\chi_{n,m}(A)=\sum_{r=0}^{[n/2]}\sum_{s=0}^{[m/2]}(n+1-2r)$$



Proof: Let I be the T_2-ideal of graded identities of A. As we said above $V_{n,0}/I_{n,0}$ is the one-dimensional S_n-module with character



while for $m > 0$, $V_{n,m}/I_{n,m} = \oplus W_q$. Hence by Lemma 4

$$\chi_{n,m}(A) = \sum_{q=0}^{n}\sum_{t=0}^{q^*}\sum_{s=0}^{[m/2]} \begin{array}{l} \leftarrow\ n-t\ \rightarrow \\ \boxed{} \\ \boxed{} \\ \quad \leftarrow t \rightarrow \end{array} \otimes \begin{array}{l} \leftarrow\ m-s\ \rightarrow \\ \boxed{} \\ \boxed{} \\ \quad \leftarrow s \rightarrow \end{array}$$

where $q^* = \min\{q, n-q\}$.

Now for a fixed r, with $0 \le r \le [n/2]$, there exist precisely $(n+1-2r)$ values of q, with $0 \le q \le n$, such that $r \le \min\{q, n-q\}$, namely $q = r, r+1, \dots, n-r$.

Thus, for $m > 0$, one has

$$\chi_{n,m}(A) = \sum_{r=0}^{[n/2]}\sum_{s=0}^{[m/2]} (n+1-2r) \begin{array}{l} \leftarrow\ n-r\ \rightarrow \\ \boxed{} \\ \boxed{} \\ \quad \leftarrow r \rightarrow \end{array} \otimes \begin{array}{l} \leftarrow\ m-s\ \rightarrow \\ \boxed{} \\ \boxed{} \\ \quad \leftarrow s \rightarrow \end{array}$$

∎

The graded identities of $M_{1,1}(E)$

First we recall some results of [4] in order to prove our main result about $M_{1,1}(E)$.

Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded algebra, and let I_1 be the T_2-ideal of its graded identities. In [4, Lemmas 1, 4] it is proved that the T_2-ideal I_2 associated to the $\mathbb{Z}_2$-graded algebra $B = A_0 \otimes E_0 + A_1 \otimes E_1$ satisfies the condition $I_2 = I_1^*$, where $*$ is defined as in [4, p.362].

Moreover, for all $n, m \ge 0$, one has $I_2 \cap V_{n,m} = (I_1 \cap V_{n,m})^*$, and $*$ acts linearly on $V_{n,m}$ in the following way:

Let $M = M(y_1, \dots, y_n, z_1, \dots, z_m)$ be a monomial in $V_{n,m}$. Denote the order in which the z_i's occur in M by $z_{i_1}, z_{i_2}, \dots, z_{i_m}$. Then $M^* = (-1)^\pi M$ where π is the permutation $\begin{pmatrix} 1 \cdots m \\ i_1 \cdots i_m \end{pmatrix}$.

We have

LEMMA 6: *Let Δ be the one-dimensional $S_n \times S_m$-module which affords the $S_n \times S_m$-representation given by $(\sigma, \tau) \longmapsto (-1)^\tau$. Then for every $S_n \times S_m$ -submodule N of $V_{n,m}$ we have*

(i) *N^* is a $S_n \times S_m$-submodule of $V_{n,m}$,*

(ii) $N^* \cong N \otimes_F \Delta$.

Proof: By the definition of the action of $*$ on $V_{n,m}$ it follows that $((\sigma,\tau)M)^* = (-1)^\tau(\sigma,\tau) \cdot M^*$, for all $(\sigma,\tau) \in S_n \times S_m$ and for all monomials M of $V_{n,m}$. Since $*$ acts linearly on $V_{n,m}$ this implies the result. ∎

As a consequence we obtain (see also [2, Lemma 6])

LEMMA 7: *Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded algebra, and let $B = A_0 \otimes E_0 + A_1 \otimes E_1$ the graded tensor product of A with E.*

If $\chi_{n,m}(A) = \sum m_{\lambda,\mu}[\lambda] \otimes [\mu]$ then $\chi_{n,m}(B) = \sum m_{\lambda,\mu}[\lambda] \otimes [\mu']$, where μ' is the conjugate partition of μ.

Proof: Let I be the T_2-ideal of graded identities of A and decompose $V_{n,m}$ as the direct sum of the $S_n \times S_m$-submodules $I_{n,m}$, N.

Since $*$ acts linearly on $V_{n,m}$ we have $V_{n,m} = (I_{n,m})^* \oplus N^*$. As we said above, $(I_{n,m})^* = (I \cap V_{n,m})^* = I^* \cap V_{n,m}$, hence by Lemma 6 and [4, Lemma 4] we have

$$\begin{aligned}\chi_{n,m}(B) &= \chi_{n,m}(N^*) = \chi_{n,m}(N \otimes_F \Delta) \\ &= (\sum m_{\lambda,\mu}[\lambda] \otimes [\mu]) \otimes ([(n)] \otimes [(-1)^m]) = \sum m_{\lambda,\mu}[\lambda] \otimes [\mu']\end{aligned}$$

by 6.6 of [3]. ∎

We are ready to prove

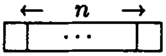
THEOREM 1: *Let J be the T_2-ideal of graded identities of $M_{1,1}(E)$, then*

(1) *J is generated by $y_1y_2 - y_2y_1$ and $z_1z_2z_3 + z_3z_2z_1$,*

(2) *$c_{n,0}(J) = 1$ and $c_{n,m}(J) = 2^n \binom{m}{[\frac{m}{2}]}$ for $n \geq 0$, $m > 0$,*

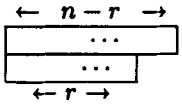
(3)

$$\chi_{n,0}(J) =$$

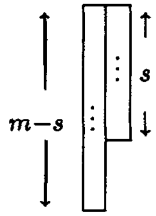


and, for $m > 0$,

$$\chi_{n,m}(J) = \sum_{r=0}^{[n/2]} \sum_{s=0}^{[m/2]} (n+1-2r)$$



$\otimes$



Proof: Let $A = M_2(F)$ with the non-trivial grading A_0, A_1 where

$$A_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a, d \in F \right\}, \quad A_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \middle| b, c \in F \right\};$$

then $M_{1,1}(E)$ is isomorphic to the graded tensor product $A_0 \otimes E_0 + A_1 \otimes E_1$.

Therefore (2) and (3) follow from Lemmas 5 and 7. Moreover by Lemma 4 of [4] $J = I^*$, where I is the T_2-ideal of graded identities of A. By definition, I^* is the T_2-ideal generated by all multilinear polynomials f^*, with $f \in I$. By Lemma 2, I is generated by $y_1y_2 - y_2y_1$ and $z_1z_2z_3 - z_3z_2z_1$; hence, as $\mathrm{char} F = 0$, any multilinear polynomial f of I is a linear combination of the polynomials $a_0(a_1a_2 - a_2a_1)a_3$ and $b_0(b_1b_2b_3 - b_3b_2b_1)b_4$, where a_i, b_j are multilinear monomials of $F\{X\}$ and moreover $a_i \in \mathcal{F}_0$, $b_j \in \mathcal{F}_1$ for $i = 1,2$ and $j = 1,2,3$.

We remark that if a, b are monomials of $F\{X\}$, linear in the disjoint ordered subsets $\{z_{i_1}, \ldots, z_{i_r}\}$, $\{z_{j_1}, \ldots, z_{j_s}\}$ of Z, then $(ab)^* = (-1)^\sigma a^*b^*$, where σ is the permutation $\binom{t_1 \cdots t_r t_{r+1} \cdots t_{r+s}}{i_1 \cdots i_r j_1 \cdots\cdots j_s}$ and $t_1, t_2, \ldots, t_{r+s}$ are the integers $i_1, \ldots, i_r$, $j_1, \ldots, j_s$ written in increasing order.

Moreover, if r and s are both even then the permutations $\binom{t_1 \cdots t_r t_{r+1} \cdots t_{r+s}}{i_1 \cdots i_r j_1 \cdots\cdots j_s}$ and $\binom{t_1 \cdots t_s t_{s+1} \cdots t_{r+s}}{j_1 \cdots j_s i_1 \cdots\cdots i_r}$ have the same sign while if r, s, u are odd then the permutations

$$\begin{pmatrix} t_1 \cdots t_r t_{r+1} \cdots t_{r+s} t_{r+s+1} \cdots t_{r+s+u} \\ i_1 \cdots i_r v_1 \cdots\cdots v_s j_1 \cdots\cdots\cdots j_u \end{pmatrix}$$

and

$$\begin{pmatrix} t_1 \cdots t_u t_{u+1} \cdots t_{u+s} t_{u+s+1} \cdots t_{u+s+r} \\ j_1 \cdots j_u v_1 \cdots\cdots v_s i_1 \cdots\cdots\cdots i_r \end{pmatrix}$$

have opposite sign.

This implies that $(a_0(a_1a_2 - a_2a_1)a_3)^* = \pm a_o^*(a_1^*a_2^* - a_2^*a_1^*)a_3^*$ and also

$$(b_0(b_1b_2b_3 - b_3b_2b_1)b_4)^* = \pm b_0^*(b_1^*b_2^*b_3^* - b_3^*b_2^*b_1^*)b_4^*$$

which are both in the T_2-ideal generated by $y_1y_2 - y_2y_1$ and $z_1z_2z_3 + z_3z_2z_1$. This completes the proof. ∎

For the next result we need to recall the following definition:

Two P.I.-algebras A, B are equivalent if they satisfy the same polynomial identities.

As a consequence of Theorem 1 we have

THEOREM 2: *$M_{1,1}(E)$ and $E \otimes E$ are equivalent.*

Proof: Let P and Q be the T-ideals of the polynomial identities of $M_{1,1}(E)$ and $E \otimes E$ respectively. By a result of Popov [5], Q is generated by $[x_1, x_2, [x_3, x_4], x_5]$ and $[[x_1, x_2], x_2^2]$. Since these polynomials are identities of $M_{1,1}(E)$ then $Q \subseteq P$.

In order to prove the reverse inclusion, we remark that $E \otimes E$ is a $\mathbb{Z}_2$-graded algebra with grading $A_0 = E_0 \otimes E_0 + E_1 \otimes E_1$, $A_1 = E_0 \otimes E_1 + E_1 \otimes E_0$. With respect to this grading, the polynomials $y_1y_2 - y_2y_1$ and $z_1z_2z_3 + z_3z_2z_1$ are graded identities of $E \otimes E$.

Therefore, it follows from Theorem 1 that any graded identity of $M_{1,1}(E)$ is also a graded identity of $E \otimes E$.

Now, let V_n be the space of all multilinear polynomials of degree n in $x_1, \dots, x_n$ and $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded algebra.

Let S, T be a partition of $\{1, 2, \dots, n\}$, then, as in [2], $I_{S,T}(A)$ will be the subspace of all $f(x_1, \dots, x_n) \in V_n$ which vanish under all substitutions $x_i \longmapsto a_i$, with $a_i \in A_0$ whenever $i \in S$ and $a_i \in A_1$ whenever $i \in T$.

As quoted in [2], since $A = A_0 + A_1$, a multilinear polynomial $f(x_1, \dots, x_n)$ will be an identity for A if and only if it vanishes under every homogeneous substitution; that is f is an identity for A if and only if $f(x_1, \dots, x_n) \in I_{S,T}(A)$ for every $\{S, T\}$ partition of $\{1, 2, \dots, n\}$.

As we showed above $I_{S,T}(M_{1,1}(E)) \subseteq I_{S,T}(E \otimes E)$, hence we must have $P_n \subseteq Q_n$ for all $n \geq 0$. Since char$F = 0$ then $P \subseteq Q$ and we are done. ∎

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